QUANTUM AND CLASSICAL PHASE TRANSITIONS: SOME ELEMENTARY REMARKS

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Reference and more details: Quantum Field Theory and Critical Phenomena, Oxford Univ. Press 2002

Contents

ABSTRACT

In these lectures, we explain with simple standard examples, the transition between classical and quantum statistics at equilibrium.

First, we discuss the non-relativistic particle. Though the example is somewhat trivial, two generic phenomena emerge: the role of the thermal wave length and the property of dimensional reduction between quantum and classical statistics from the path integral viewpoint.

Then, we examine a quantum Bose gas, a model for the Helium phase transition where the transition between quantum and classical physics is more subtle.

Finally, we study a quantum field theory that describes the quantum scalar particle in four dimensions but in lower dimensions can be mapped onto classical spin models. In this mapping, the quantum system at finite (non-zero) temperature is mapped onto the classical model with a finite size in one space direction. Finite size analysis can then be applied.

As a mathematical tool, functional integration will be used throughout.

Lecture 1: THE NON-RELATIVISTIC PARTICLE AT EQUILIBRIUM IN THE SEMI-CLASSICAL LIMIT

1.1 The non-relativistic particle at thermal equilibrium

For a one-particle, one-dimensional quantum Hamiltonian of the form

$$\mathbf{H} = \mathbf{p}^2/2m + V(\mathbf{q}),$$

where $V(q)$ is a smooth function, the partition function is given by the integral over paths in imaginary time, $\beta = 1/T$

$$\mathcal{Z}(\beta) = \operatorname{tr} \mathrm{e}^{-\beta \mathbf{H}} = \int [\mathrm{d}q(t)] \exp\left[-\mathcal{S}(q)\right],$$

where the paths satisfy periodic boundary conditions: $q(0) = q(\beta)$ and

$$\mathcal{S}(q) = \int_0^\beta \mathrm{d}t \left[\tfrac{1}{2} m \dot{q}^2(t)/\hbar^2 + V\big(q(t)\big)\right]. \tag{1.1}$$

Note that $\hbar$ appears only in front of the kinetic term.

1.2 The semi-classical expansion

We study the limit $\hbar \to 0$ at fixed temperature of the partition function. The quantum partition function converges toward the classical partition function but we also calculate the first correction in the semi-classical expansion.

The explicit calculation will help us identifying the expansion parameter, since $\hbar$ has a dimension and, thus, must be divided by a quantity that has the dimension of an action.

We infer that classical limit and finite temperature effects are related.

In the action (1.1) , the leading term for $\hbar \to 0$ is the kinetic term. The leading trajectories thus are those that satisfy $\dot{q} = 0$ and correspond to constant functions.

From the viewpoint of the steepest descent method, one finds a one-parameter family of degenerate saddle points.

1.2.1 Explicit calculation

To be able to sum over all saddle points, we first evaluate the diagonal matrix element

$$\langle q_0 | \, \mathrm{e}^{-\beta H} \, | q_0 \rangle = \int_{q(0)=q(\beta)=q_0} [\mathrm{d}q(t)] \exp\left[-\mathcal{S}(q)\right],$$

to which only one saddle point, $q(t) \equiv q_0$, contributes. After the translation $q(t) \mapsto q(t) + q_0$, the path integral becomes

$$\langle q_0 | \, \mathrm{e}^{-\beta H} \, | q_0 \rangle = \int_{q(0)=q(\beta)=0} [\mathrm{d}q(t)] \exp\left[-\Sigma(q)\right] \tag{1.2}$$

with

$$\Sigma(q) = \int_0^\beta \mathrm{d}t \left[\tfrac{1}{2} m \dot{q}^2(t)/\hbar^2 + V\big(q_0 + q(t)\big)\right].$$

The form of the action shows that $\dot{q}$ is formally of order $\hbar$. With the boundary conditions in (1.2), the saddle point is $q(t) \equiv 0$ and thus $q(t)$ itself is of order $\hbar$.

The potential can be expanded in powers of $q(t)$,

$$V\big(q_0 + q(t)\big) = V(q_0) + V'(q_0)q(t) + \tfrac{1}{2}V''(q_0)q^2(t) + O(\hbar^3),$$

as well as the integrand in the integral (1.2). Each term has the form of a Gaussian expectation value:

$$\langle q_0|\,\mathrm{e}^{-\beta H}\,|q_0\rangle = \mathcal{N}(\beta)\,\mathrm{e}^{-\beta V(q_0)}\left[1 - V'(q_0)\int_0^\beta \mathrm{d}t\,\langle q(t)\rangle_0 \right.$$
$$\left. + \tfrac{1}{2}(V'(q_0))^2\int_0^\beta \mathrm{d}t\,\mathrm{d}u\,\langle q(t)q(u)\rangle_0 - \tfrac{1}{2}V''(q_0)\int_0^\beta \mathrm{d}t\,\left\langle q^2(t)\right\rangle_0 + O(\hbar^3)\right],$$

where $\langle\bullet\rangle_0$ refers to the expectation value with respect to the measure corresponding to the free action $\mathcal{S}_0(q) = \frac{1}{2}m\int \mathrm{d}t\,\dot{q}^2(t)/\hbar^2$.

These Gaussian expectation values are determined by the two-point function $\Delta(t,u)$ corresponding to the free action with the boundary conditions $q(0) = q(\beta) = 0$:

$$\langle q_0 | \, \mathrm{e}^{-\beta H} \, | q_0 \rangle = \mathcal{N}(\beta) \, \mathrm{e}^{-\beta V(q_0)} \left[1 + \frac{\hbar^2}{2m} (V'(q_0))^2 \int_0^\beta \Delta(t,u) \mathrm{d}t \, \mathrm{d}u \right. \\ \left. - \frac{\hbar^2}{2m} V''(q_0) \int_0^\beta \Delta(t,t) \mathrm{d}t + O(\hbar^3) \right] .$$

The normalization is given by

$$\mathcal{N}(\beta) = \langle q = 0 | \, \mathrm{e}^{-\beta p^2/2m} \, | q = 0 \rangle = \int \frac{\mathrm{d}p}{2\pi\hbar} \, \mathrm{e}^{-\beta p^2/2m} = \frac{1}{\hbar} \sqrt{\frac{m}{2\pi\beta}} . \qquad (1.3)$$

One then infers that $\Delta(t,u) = \Delta(u,t)$ is the solution of the equation

$$-\ddot{\Delta}(t,u) = \delta(t-u) \quad \text{with} \quad \Delta(0,u) = \Delta(\beta,u) = 0 \, .$$

The solution is

$$\Delta(t,u) = -\tfrac{1}{2}|t-u| + \tfrac{1}{2}(t+u-2ut/\beta).$$

Using the two-point function, we evaluate the expectation values explicitly:

$$\langle q_0|\, \mathrm{e}^{-\beta H}\, |q_0\rangle = \mathcal{N}(\beta)\, \mathrm{e}^{-\beta V(q_0)} \bigg[1 + \frac{\hbar^2\beta^3}{24m}(V'(q_0))^2 - \frac{\hbar^2\beta^2}{12m} V''(q_0) \\ + O(\hbar^3) \bigg]. \tag{1.4}$$

1.2.2 The partition function

The partition function is obtained by integrating over q_0:

$$\mathcal{Z}(\beta) = \int \mathrm{d}q \, \langle q | \, \mathrm{e}^{-\beta H} \, | q \rangle$$
$$= \frac{1}{\hbar}\sqrt{\frac{m}{2\pi\beta}} \int \mathrm{d}q \, \mathrm{e}^{-\beta V(q)} \left[1 + \frac{\hbar^2\beta^3}{24m}(V'(q))^2 - \frac{\hbar^2\beta^2}{12m} V''(q) + O(\hbar^3) \right].$$

Integrating the second term by parts and exponentiating, one finds

$$\mathcal{Z}(\beta) = \frac{1}{\hbar}\sqrt{\frac{m}{2\pi\beta}} \int \mathrm{d}q \, \mathrm{e}^{-\beta V(q)} \left[1 - \frac{\hbar^2\beta^2}{24m} V''(q) + O(\hbar^3) \right]$$
$$= \frac{1}{\hbar}\sqrt{\frac{m}{2\pi\beta}} \int \mathrm{d}q \, \exp\left[-\beta V_{\mathrm{eff.}}(q) \right]$$

with

$$V_{\mathrm{eff.}}(q) = V(q) + \frac{\beta\hbar^2}{24m} V''(q) + O(\hbar^3). \tag{1.5}$$

1.2.3 Discussion

(i) The leading contribution for $\hbar \to 0$ is the classical partition function corresponding to the reduced Boltzmann weight obtained by integrating the Boltzmann weight $\mathrm{e}^{-\beta H(p,q)}$ over the momentum p, where H is the classical Hamiltonian in a phase space normalized with respect to $2\pi\hbar$.

Indeed, for a Hamiltonian $H = p^2/2m + V(q)$, the two forms of $\mathcal{N}(\beta)$ in equations (1.3), lead to the identity

$$\mathcal{Z}_{\mathrm{cl.}}(\beta) = \int \frac{\mathrm{d}p\mathrm{d}q}{2\pi\hbar}\, \mathrm{e}^{-\beta H(p,q)} = \frac{1}{\hbar}\sqrt{\frac{m}{2\pi\beta}} \int \mathrm{d}q\; \mathrm{e}^{-\beta V(q)}\,. \tag{1.6}$$

(ii) The first relative correction in the integrand is

$$\frac{\beta\hbar^2 V''(q)}{24mV(q)} = \lambda^2_{\mathrm{th.}} \frac{V''(q)}{24V(q)}$$

where $\lambda_{\mathrm{th.}} = \hbar\sqrt{\beta/m}$ is the thermal wavelength.

Introducing $l_{\text{pot.}} = \sqrt{|\langle V(q)\rangle / \langle V''(q)\rangle|}$, a length scale typical of the space variations of the potential (which we assume to be characterized by only one length scale), one notes that the ratio between the classical term and the first quantum correction can be characterized by the ratio $\lambda_{\text{th.}}/l_{\text{pot.}}$.

At high temperature $T = 1/\beta$, and thus small β, the thermal wavelength is small and the statistical behaviour classical. On the contrary, at low temperature quantum effects become dominant.

(iii) Formally, the classical limit corresponds to a kind of dimensional reduction. The quantum partition function involves an integration over paths, that is, one-dimensional objects. By contrast, the classical function, involves only an integration over a zero mode (in the sense of the Fourier series, see next section), which corresponds to a point, and has zero dimension.

1.3 Alternative calculation: mode expansion

An alternative calculation is based on expanding $q(t)$ as a Fourier series on an orthonormal basis of periodic functions on the interval $[0, \beta]$:

$$q(t) = q_0 + \delta q(t), \quad \delta q(t) = \sqrt{2/\beta} \sum_{n>0} \left[a_n \cos(2\pi n t/\beta) + b_n \sin(2\pi n t/\beta)\right].$$

The method has the advantage of being generalizable to field theories.

We have separated the mode q_0 because it gives no contribution to the time derivative and, thus, is not sensitive to the limit $\hbar \to 0$. In contrast, $\delta q(t)$ and thus the coefficients a_n and b_n are formally of order $\hbar$. This justifies expanding the potential in powers of $\delta q(t)$:

$$V\big(q(t)\big) = V(q_0) + \delta q(t) V'(q_0) + \tfrac{1}{2}\big(\delta q(t)\big)^2 V''(q_0) + O\left((\delta q)^3\right).$$

We then integrate over t and use the orthogonality of the basis. As a consequence, the term of order δq vanishes. The action then becomes

$$\mathcal{S}(q) = \beta V(q_0) + \sum_{n>0} (a_n^2 + b_n^2) \left[\frac{2mn^2\pi^2}{\beta^2\hbar^2} + \frac{1}{2} V''(q_0) \right] + O(a_n^3, a_n^2 b_n, \ldots).$$

One can then replace the integration over paths $q(t)$ by an integration over the coefficients of the expansion of $q(t)$ on the orthonormal basis, q_0 (this mode is not normalized, but the Jacobian is a constant), and on $\{a_n, b_n\}$. We normalize all integrals by dividing them by the integral for $V''(q) \equiv 0$. Then,

$$\begin{aligned} \int \mathrm{d}a_n \exp\left[-\left(\frac{2mn^2\pi^2}{\beta^2\hbar^2} + \frac{1}{2} V''(q_0) \right) a_n^2 \right] &\propto \left[1 + \frac{\beta^2\hbar^2}{4m\pi^2 n^2} V''(q_0) \right]^{-1/2} \\ &\sim \exp\left[-\frac{\beta^2\hbar^2}{8m\pi^2 n^2} V''(q_0) \right]. \end{aligned}$$

Summing over n (and using $\sum_n 1/n^2 = \pi^2/6$), one finds the partition function, up to a normalization.

The normalization $\mathcal{N}(\beta)$, which is independent of the potential, can be determined by noting that before integration over q_0 one has obtained a contribution to a diagonal element of the density matrix of the form $\langle q| \, \mathrm{e}^{-\beta H} \, |q\rangle$. For $V \equiv 0$, it is given by expression (1.3). One then recovers the result (1.5).

Lecture 2: THE BOSE GAS: FROM BOSE–EINSTEIN CONDENSATION TO THE HELIUM PHASE TRANSITION

2.1 The weakly interacting Bose gas

The partition function of a gas of identical bosons of mass m, at temperature $T = 1/\beta$, can be expressed as an integral

$$\mathcal{Z} = \int [\mathrm{d}\psi(t,x)\mathrm{d}\bar{\psi}(t,x)]\, \mathrm{e}^{-\mathcal{S}(\bar{\psi},\psi)},$$

over fields $\bar{\psi}, \psi$ periodic in Euclidean time,

$$\psi(0,x) = \psi(\beta,x), \quad \bar{\psi}(0,x) = \bar{\psi}(\beta,x),$$

associated with boson creation and annihilation.

We assume that the gas is sufficiently dilute for the two-body interactions to be weak and three-body or higher interactions to be totally negligible.

2.1.1 Euclidean quantum action

Since one is interested only in long wavelength phenomena, the two-body potential itself can be replaced by a delta-function and parametrized in terms of the s-wave scattering length $a > 0$ (because the interaction is assumed short range and repulsive).

For $d = 3$, the effective Euclidean action of the system may then be written as

$$\mathcal{S}(\bar{\psi}, \psi) = \int_0^{1/T} \mathrm{d}t \int \mathrm{d}^3x \left\{ -\bar{\psi}(t, x) \left(\frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \nabla_x^2 + \mu \right) \psi(t, x) \right. \\ \left. + \frac{2\pi\hbar^2 a}{m} \left[\bar{\psi}(t, x)\psi(t, x) \right]^2 \right\}, \tag{2.1}$$

where μ is the chemical potential.

The condition that the interaction is weak implies that $a \ll \bar{\lambda}$, where $\bar{\lambda}$ is the thermal wavelength,

$$\bar{\lambda} = \hbar\sqrt{2\pi/mT}\,.$$

2.1.2 Equation of state and two-point function

Quite generally, the equation of state can be expressed in terms of the $\langle \bar{\psi}\psi \rangle$ correlation function G. The density ρ, in d space dimensions, assuming translation invariant boundary conditions, is given by

$$\rho = T \frac{\partial \ln \mathcal{Z}}{\partial \mu} = \langle \bar{\psi}(0,0)\psi(0,0) \rangle = \int \frac{\mathrm{d}^d p}{(2\pi)^d} \sum_{\nu \in \mathbb{Z}} \tilde{G}(p, \omega_\nu; \mu),$$

where $\tilde{G}(p, \omega_\nu; \mu)$ is the two-point function in Fourier representation and the quantities,

$$\omega_\nu = 2\pi\nu T \,, \quad \nu \in \mathbb{Z} \,,$$

discrete due to the periodic boundary conditions in time, are also called Matsubara frequencies.

2.2 Independent bosons: Bose–Einstein condensation

For vanishing self-interaction (here $a = 0$), the partition function can be calculated.

It is convenient to confine the gas in a hypercubic box of linear size L with periodic boundary conditions and then study the large L regime (though a harmonic potential would be more realistic).

Then the fields $\varphi(t, \mathbf{x}), \bar{\varphi}(t, \mathbf{x})$ are periodic functions of all space variables, of period L. The arguments of the Fourier transform belong to the lattice,

$$\mathbf{p} = \frac{2\pi\hbar}{L}\mathbf{n}\,, \quad \mathbf{n} \in \mathbb{Z}^d\,.$$

The free energy, $\mathcal{W}_0(\beta = 1/T) = \beta^{-1} \ln \mathcal{Z}_0(\beta)$, is

$$\mathcal{W}_0(\beta) = -\frac{1}{\beta} \sum_{\mathbf{n} \in \mathbb{Z}^d} \ln\left(1 - \mathrm{e}^{-\beta(p^2/2m - \mu)}\right).$$

For $L \gg \not\lambda$ large, the sum over momentum modes can be replaced by an integral (a semi-classical approximation) and the free energy per unit volume, which is the pressure Π, becomes ($\mathrm{d}\mathbf{n} = \mathrm{d}\mathbf{p}L/2\pi\hbar$)

$$\Pi = L^{-d}\mathcal{W}_0(\beta) = -\frac{1}{\beta}\int \frac{\mathrm{d}^d p}{(2\pi\hbar)^d} \ln\left(1 - \mathrm{e}^{-\beta(p^2/2m-\mu)}\right).$$

The Bose gas is stable only if the chemical potential is non-positive.

Taking the derivative of $\ln \mathcal{Z}$ with respect to $\beta\mu$ (β fixed), one obtains the average particle number and thus the gas density,

$$\rho = L^{-d}\langle N\rangle = \frac{1}{L^d\beta}\int \mathrm{d}t\,\mathrm{d}^d x\,\langle\varphi(t,x)\bar{\varphi}(t,x)\rangle = \frac{1}{(2\pi\hbar)^d}\int \frac{\mathrm{d}^d p}{\mathrm{e}^{\beta(p^2/2m-\mu)}-1}\,. \tag{2.2}$$

The equation of state (2.2) exhibits the phenomenon of Bose–Einstein condensation.

At fixed temperature $T = 1/\beta$, the density ρ is an increasing function of μ. For space dimensions $d > 2$, since $\mu \le 0$, ρ is bounded by the value ρ_c of the integral calculated for $\mu = 0$:

$$\rho \le \rho_c = \frac{1}{(2\pi\hbar)^d} \int \frac{\mathrm{d}^d p}{\mathrm{e}^{\beta p^2/2m} - 1} = \zeta(d/2) \left(\frac{mT}{2\pi\hbar^2} \right)^{d/2} = \zeta(d/2) \lambda\!\!\!^{-}{}^{-d},$$

($\zeta(s)$ is Riemann's ζ-function). Conversely, at fixed density, the equation of state has a solution up to the minimal temperature

$$T_c = \frac{2\pi\hbar^2}{m} \left(\frac{\rho}{\zeta(d/2)} \right)^{2/d}.$$

In three dimensions,

$$T_c^0(\rho) \underset{d=3}{\propto} (\hbar^2/m)\rho^{2/3}.$$

To understand the physics below T_c, one has to return to the exact form in a finite box, where the momentum modes are discrete and the integral is replaced by a sum of poles. One then discovers that a macroscopic fraction of the free Bose gas condenses in the ground state, which here is the zero momentum mode.

In two dimensions, because ρ_c diverges, there is no condensation.

The discussion indicates that, in the limit of vanishing repulsive interactions, the phase transition of the interacting model (2.1) reduces to the Bose–Einstein condensation of the free Bose gas.

2.3 The weakly interacting Bose gas and the Helium phase transition

Phase transitions. In the interacting model, for $d > 2$ a $U(1)$ phase transition of superfluid Helium type occurs at a critical chemical potential μ_c where $\tilde{G}^{-1}(p = 0, \omega = 0; \mu_c) = 0$ and, thus, the correlation length, which characterizes the decay of correlations at large distance in the disordered phase, diverges.

At $d = 2$, the system exhibits the peculiar Kosterlitz–Thouless transition: below T_c the correlation length diverges but without ordering, a phase that we will not discuss here.

The theory of critical phenomena tells us that the universal properties near a continuous phase transition in systems with dimension $d \leq 4$ depends primarily on contributions from the small momenta or large distance (or IR) region.

The propagator, or leading order two-point function for $a \to 0$, in Fourier representation is

$$\tilde{G}^{-1}(p,\omega;\mu) = p^2/2m - i\omega_\nu - \mu\,, \quad \omega_\nu = 2\pi T\,.$$

The momentum pole closest to $|p| = 0$ is obtained for $\omega_\nu = 0$. One finds $|p| = \sqrt{-2m\mu} \equiv \hbar/\xi$, where ξ is the correlation length. As soon as the frequency gap T satisfies

$$T \gg |p|^2/2m = \hbar^2/m\xi^2 \;\Leftrightarrow\; \xi \gg \lambda\!\!\!/\,,$$

the problem is simplified since the large scale properties remain sensitive only to the zero-mode $\nu = 0$ corresponding to the $\omega_\nu = 0$ Matsubara frequency.

Mode expansion. Expanding

$$\psi(t,x)=\sum_{\nu\in\mathbb{Z}}\mathrm{e}^{i\omega_\nu t}\,\psi_\nu(x),\;\;\bar\psi(t,x)=\sum_{\nu\in\mathbb{Z}}\mathrm{e}^{i\omega_\nu t}\,\bar\psi_\nu(x),$$

at leading order one can thus omit the non-zero modes.

The entire calculation can be cast in terms of a classical statistical field theory.

The integration over the higher frequency modes, which are not critical, can then be done perturbatively. The effect is mainly to renormalize the coefficients of the zero-mode action.

Since at leading order we have omitted these modes, the temperature gap provides a UV large momentum cut-off $\Lambda\sim\hbar/\chi\propto\sqrt{mT}$.

In the action, we keep below the dimension $d\le 4$ arbitrary, even though we are mainly interested in $d=3$ in order to use dimensional continuation and regularization later.

From now on we set $\hbar=1$.

2.3.1 Effective classical statistical field theory

We rescale the field ψ in order to introduce more conventional field theory normalizations, and parametrize it in terms of two real fields ϕ_1, ϕ_2: $\psi_0 = \sqrt{mT}(\phi_1 + i\phi_2)$, $\bar{\psi}_0 = \sqrt{mT}(\phi_1 - i\phi_2)$. The action becomes

$$\mathcal{Z} = \int [\mathrm{d}\boldsymbol{\phi}(x)] \exp\left[-\mathcal{S}(\boldsymbol{\phi})\right]$$

with

$$\mathcal{S}(\boldsymbol{\phi}) = \int \left\{ \tfrac{1}{2}\left[\nabla_x \boldsymbol{\phi}(x)\right]^2 + \tfrac{1}{2} r \boldsymbol{\phi}^2(x) + \frac{u}{4!}\left[\boldsymbol{\phi}^2(x)\right]^2 \right\} \mathrm{d}^d x\,,$$

where $r = -2m\mu$ and, for $d = 3$, $u = 96\pi^2 a/\lambda\!\!\!{}^{-}{}^2$.

The Euclidean action reduces to the ordinary $O(2)$ symmetric $(\boldsymbol{\phi}^2)^2$ field theory, which indeed is known to also describe the universal properties of the superfluid Helium transition.

A transition occurs at a value $r = r_c$ where the correlation length diverges.

2.4 Renormalization group and universality

To gain some insight into the universal properties of the phase transition, RG methods are then required. For simplicity, we discuss here only the properties of the critical two-point correlation function.

We introduce the dimensionless coupling constant (Λ is the large momentum cut-off or inverse microscopic scale)

$$g = \Lambda^{d-4} u \propto (a/\chi)^{d-2} \ll 1 \,.$$

We denote by $\tilde{\Gamma}^{(2)}(p)$ the inverse two-point correlation function in Fourier representation. At the transition point $r = r_c$, $\tilde{\Gamma}^{(2)}(0) = 0$.

Then, $\tilde{\Gamma}^{(2)}(p)$ satisfies an RG equation of the form

$$\left(\Lambda \frac{\partial}{\partial \Lambda} + \beta(g) \frac{\partial}{\partial g} - \eta(g)\right) \tilde{\Gamma}^{(2)}(p, \Lambda, g) = 0 \,, \tag{2.3}$$

where $\beta(g)$ and $\eta(g)$ are two RG functions calculable as series in g.

2.4.1 Solution of the RG equations: The ε-expansion

Equation (2.3) can be solved by the method of characteristics. One introduces a dilatation parameter λ and looks for two functions $g(\lambda)$ and $Z(\lambda)$ such that

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda}\left[Z^{-1}(\lambda)\tilde{\Gamma}^{(2)}\big(p; g(\lambda), \lambda\Lambda\big)\right] = 0\,. \tag{2.4}$$

The function $g(\lambda)$ is the effective coupling at the scale λ.

Differentiating explicitly with respect to λ, one finds that equation (2.4) is consistent with equation (2.3) provided that

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} g(\lambda) = \beta\big(g(\lambda)\big), \qquad g(1) = g\,, \tag{2.5a}$$

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} \ln Z(\lambda) = \eta\big(g(\lambda)\big), \qquad Z(1) = 1\,. \tag{2.5b}$$

Equation (2.4) implies

$$\tilde{\Gamma}^{(2)}(p; g, \Lambda) = Z^{-1}(\lambda)\tilde{\Gamma}^{(2)}\big(p; g(\lambda), \lambda\Lambda\big).$$

It is actually convenient to rescale Λ by a factor $1/\lambda$. Using then the dimensional relation

$$\tilde{\Gamma}^{(2)}(p; g, \Lambda/\lambda) = \frac{1}{\lambda^2}\tilde{\Gamma}^{(2)}(\lambda p; g, \Lambda),$$

one can write the equation as

$$\tilde{\Gamma}^{(2)}(\lambda p; g, \Lambda) = Z^{-1}(\lambda)\lambda^2\tilde{\Gamma}^{(2)}\bigl(p; g(\lambda), \Lambda\bigr). \tag{2.6}$$

The interpretation of the equation is that it is equivalent to decrease the momentum Λ or at p fixed to vary the effective coupling.

The solution of the flow equation $(2.5a)$ for the effective coupling is

$$\int_g^{g(\lambda)} \frac{\mathrm{d}g'}{\beta(g')} = \ln\lambda\,.$$

We are interested in the regime $\lambda \to 0$. For $\beta(g) < 0$, the equation implies $g(\lambda) > g$ and for $\beta(g) > 0$ it implies $g < g(\lambda)$.

A zero of the β-function is a fixed point. A zero of the β-function with a negative slope is IR repulsive and a zero with a positive slope is IR attractive.

Perturbative calculations yield

$$\beta(g) = -(4-d)g + \tfrac{5}{24\pi^2} g^2 + \mathcal{O}(g^3), \quad \eta(g) = \tfrac{1}{18(8\pi^2)^2} g^2 + \mathcal{O}(g^3).$$

For $d < 4$, the β-function has two zeros, the zero $g = 0$ which corresponds to the IR repulsive Gaussian fixed point and, at least for $4 - d = \varepsilon > 0$ small, the non-trivial zero $g^* = 24\pi^2\varepsilon/5 + \mathcal{O}(\varepsilon^2)$ (see Fig. 2.1) which corresponds to an IR attractive fixed point.

This is the origin of Wilson–Fisher's famous ε-expansion.

For example, equation (2.3) then implies

$$\tilde{\Gamma}^{(2)}(p) \underset{p\to 0}{\propto} p^{2-\eta} \text{ with } \eta \equiv \eta(g^*) > 0\,.$$

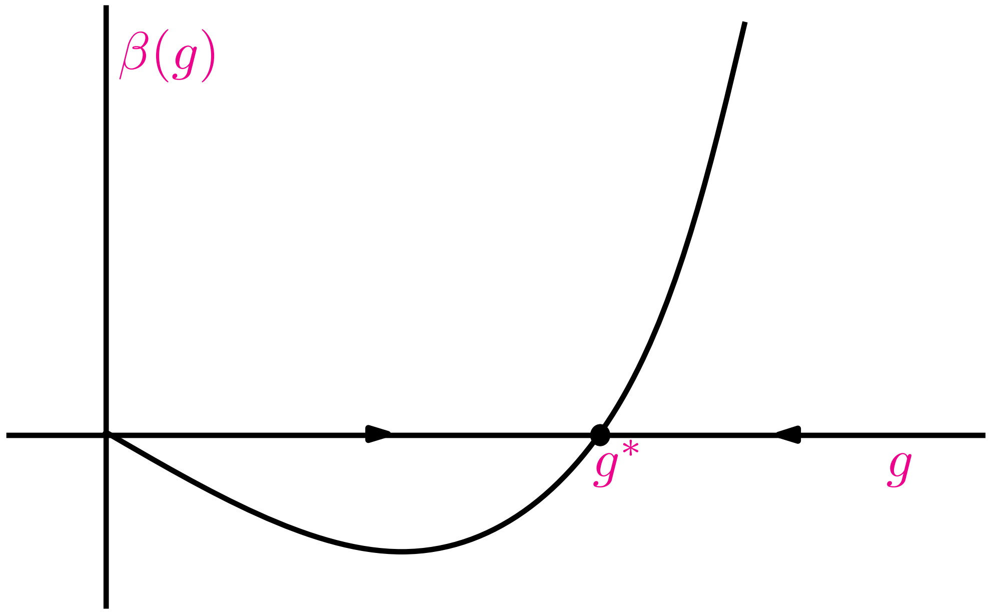

Fig. 2.1 – RG β-function and RG flow for $d < 4$.

More generally, all universal quantities like the correlation exponent ν, the scaling equation of state...., can be calculated as ε-expansions. Calculations can also be performed at fixed dimension $d = 3$ but require an additional continuity assumption.

2.4.2 RG equation: Other form of the solution

Combined with dimensional analysis, the RG equation implies that $\tilde{\Gamma}^{(2)}$ has the general form

$$\tilde{\Gamma}^{(2)}(p, \Lambda, g) = p^2 Z(g) F\big(p/\Lambda(g)\big)$$

with

$$\beta(g)\frac{\partial \ln Z(g)}{\partial g} = \eta(g), \quad \beta(g)\frac{\partial \ln \Lambda(g)}{\partial g} = -1\,.$$

Therefore, (with the normalization $Z(0) = 1$)

$$Z(g) = \exp \int_0^g \frac{\eta(g')}{\beta(g')} \mathrm{d}g' = 1 + \mathcal{O}(g^2).$$

On dimensional grounds $\Lambda(g)$ is proportional to Λ. The function $\Lambda(g)$ is then obtained by integration:

$$\Lambda(g) = g^{1/(4-d)} \Lambda \exp\left[-\int_0^g \mathrm{d}g' \left(\frac{1}{\beta(g')} + \frac{1}{(4-d)g'}\right)\right].$$

The universal large distance behaviour corresponds to momenta $|p| \ll \Lambda(g)$. In a generic situation, $g = O(1)$, then $\Lambda(g) = O(\Lambda)$. As a consequence only the universal large distance regime can be observed.

By contrast, here we are interested in $g \ll 1$, which implies $\Lambda(g) \sim g^{1/(4-d)}\Lambda \ll \Lambda$.

The scale $\Lambda(g)$ then becomes a crossover scale separating a universal long-distance regime governed by the non-trivial zero $g^* > 0$ of the β-function, from a universal short distance regime governed by the Gaussian fixed point, $g = 0$.

In this situation,

$\tilde{\Gamma}^{(2)}(p) \propto p^{2-\eta}$ for $p \ll \Lambda(g)$ and $\tilde{\Gamma}^{(2)}(p) \propto p^2$ for $\Lambda(g) \ll p \ll \Lambda$. At short distance, but much larger than the microscopic scale, one observes a Bose–Einstein condensate, while at large distance, the system looks like a Helium system.

2.5 Application: the critical temperature shift for weak interaction

The effect of a weak repulsive two-body interaction on the transition temperature of a dilute gas Bose gas at fixed density has been controversial for a long time because it was not fully realized that the effect is non-perturbative in nature.

Renormalization group then allows proving that T_c increases linearly with the strength of the interaction, parametrized in terms of the s-wave scattering length a. Moreover, when $\Delta T_c/T_c$ is expressed in terms of the dimensionless product $an^{1/3}$ (n is the density), the coefficient is universal. However, the coefficient cannot be obtained from perturbative calculations.

Recognizing that the Hamiltonian of the system, which also describes the Helium superfluid transition, is the $N = 2$ example of the general N vector model, one generalizes the problem to arbitrary N. The coefficient of $\Delta T_c/T_c$ can then be expanded in powers of $1/N$.

2.5.1 The variation of the equation of state

The shift of the critical temperature for weak coupling can be derived from the leading order non-trivial contribution at criticality (in the massless theory) to the equation of state,

$$\rho_\psi = \langle \bar{\psi}\psi \rangle = 2mT \langle \phi_1 \phi_1 \rangle = 2mT\rho\,, \quad \rho = \int^\Lambda \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{1}{\tilde{\Gamma}^{(2)}(p)},$$

where the vertex function $\tilde{\Gamma}^{(2)}$ is the inverse of the two-point function $\langle \phi_1 \phi_1 \rangle = \langle \phi_2 \phi_2 \rangle$ and the large momentum cut-off $\Lambda \sim 1/\lambda\!\!\!/$.

Because the interactions are weak, one may imagine calculating the change in the transition temperature by perturbation theory. However, the perturbative expansion for a critical theory does not exist for any fixed dimension $d < 4$.

Then,

$$\delta\rho \underset{g\to 0}{\sim} \int^\Lambda \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{p^2} \left(\frac{1}{F(p/\Lambda(g))} - 1 \right).$$

We now show that the condition $\Lambda(g) \ll \Lambda$ implies $\Delta T_c \propto \Lambda(g)$.

First, from perturbation theory one infers that for $d = 3$, the function $F(p)$ behaves for large p as (see Fig. 2.2)

$$F(p) = \tilde{\Gamma}^{(2)}(p)/p^2 = 1 + \mathcal{O}(\ln p/p^2).$$

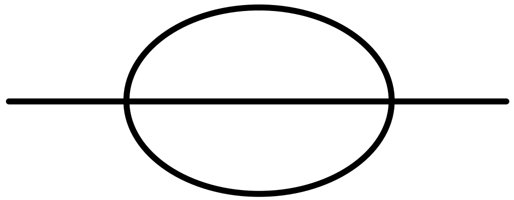

Fig. 2.2 – Leading large-momentum contribution to $\tilde{\Gamma}^{(2)}(p)$.

Therefore, the first correction to the density is convergent at large momentum and independent of the cut-off procedure:

$$\delta\rho = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{p^2} \left(\frac{1}{F(p/\Lambda(g))} - 1 \right).$$

Similarly, the IR behaviour $\tilde{\Gamma}^{(2)}(p) \propto p^{2-\eta} \Rightarrow F(p) \propto p^{-\eta}$ implies that this integral is IR convergent.

Setting $p = \Lambda(g)k$, one then finds the general form

$$\delta\rho = \Lambda(g) \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{k^2} \left(\frac{1}{F(k)} - 1 \right).$$

The g dependence is entirely contained in $\Lambda(g)$. For g small, since $\Lambda(g) \sim \Lambda g$ and $\rho \propto \Lambda \sim 1/\lambda\!\!\!/$, one concludes

$$\frac{\delta\rho}{\rho} \propto g \sim (-3/2c_0) a n^{1/3},$$

a linear behaviour that, however, is non-perturbative!

Moreover, the amplitude $-3c_0/2$ is universal.

2.5.2 The N-vector model. The large N expansion at order $1/N$

Since the function $F(p)$ cannot be evaluated from a perturbative calculation, we consider the $O(N)$ symmetric generalization of the model corresponding to the effective action. Indeed, such a generalization provides us with a tool, the large N-expansion, which allows calculating at the critical point.

The field $\boldsymbol{\phi}(x)$ then has N real components ϕ_i. The large N limit is taken at Nu fixed.

We thus consider the partition function

$$\mathcal{Z} = \int [\mathrm{d}\boldsymbol{\phi}(x)] \exp\left[-\mathcal{S}(\boldsymbol{\phi})\right],$$

where $S(\boldsymbol{\phi})$ is the $O(N)$ symmetric action

$$\mathcal{S}(\boldsymbol{\phi}) = \int \left\{ \frac{1}{2} \left[\nabla_x \boldsymbol{\phi}(x)\right]^2 + \frac{1}{2} r \boldsymbol{\phi}^2(x) + \frac{u}{4!} \left[\boldsymbol{\phi}^2(x)\right]^2 \right\} \mathrm{d}^d x \,.$$

The basic idea of the large N expansion is the same as mean field theory: for N large, the $O(N)$ invariant quantities self-average and thus have small fluctuations. This argument suggests to take $\boldsymbol{\phi}^2(x)$ as a dynamical variable. Technically, in the case of the $(\boldsymbol{\phi}^2)^2$ theory, this can be achieved by using an identity similar to the Hubbard transformation:

$$\exp\left\{-\int \mathrm{d}^d x \left[\frac{1}{2} r \boldsymbol{\phi}^2(x) + \frac{u}{4!}\left(\boldsymbol{\phi}^2\right)^2\right]\right\}$$
$$\propto \int [\mathrm{d}\lambda(x)] \exp\left[\int \mathrm{d}^d x \left(\frac{3}{2u}\lambda^2 - \frac{3r}{u}\lambda - \frac{1}{2}\lambda\boldsymbol{\phi}^2\right)\right],$$

where the integration contour is parallel to the imaginary axis. This identity allows us to rewrite the interaction term in the field integral. The new field integral is then Gaussian in $\boldsymbol{\phi}$ and the integral over the field $\boldsymbol{\phi}$ can be performed. The dependence of the partition function on N then becomes explicit.

Actually, it is convenient to separate the components of $\boldsymbol{\phi}$ into one component σ and $(N-1)$ components $\boldsymbol{\pi}$, and integrate only over $\boldsymbol{\pi}$. For N large the difference is negligible. For each $\boldsymbol{\pi}$-component, the integration yields

$$\int [\mathrm{d}\pi(x)] \exp\left\{-\tfrac{1}{2}\int \mathrm{d}^d x \left[\left(\nabla_x \pi(x)\right)^2 + \lambda(x)\pi^2(x)\right]\right\}$$
$$\propto \det{}^{-1/2}[-\nabla_x^2 + \lambda(x)] = \exp\left\{-\tfrac{1}{2}\operatorname{tr}\ln[-\nabla_x^2 + \lambda(x)]\right\} ,$$

where the general identity $\ln\det = \operatorname{tr}\ln$ has been used.

The partition function then is given by the integral

$$\mathcal{Z} = \int [\mathrm{d}\lambda(x)]\,[\mathrm{d}\sigma(x)] \exp\left[-\mathcal{S}_N(\lambda,\sigma)\right],$$

with

$$\mathcal{S}_N(\lambda,\sigma) = \int \left[\frac{1}{2}\left(\nabla_x\sigma\right)^2 - \frac{3}{2u}\lambda^2(x) + \frac{3r}{u}\lambda(x) + \frac{1}{2}\lambda(x)\sigma^2(x)\right]\mathrm{d}^d x$$
$$+ \frac{(N-1)}{2}\operatorname{tr}\ln\left[-\nabla_x^2 + \lambda(x)\right].$$

$$\mathcal{S}_N(\lambda,\sigma)=\int\left[\frac{1}{2}\left(\nabla_x\sigma\right)^2-\frac{3N}{2Nu}\lambda^2(x)+\frac{3Nr}{Nu}\lambda(x)+\frac{1}{2}\lambda(x)\sigma^2(x)\right]\mathrm{d}^d x$$
$$+\frac{(N-1)}{2}\operatorname{tr}\ln\left[-\nabla_x^2+\lambda(x)\right].$$

One then takes the large N limit at Nu fixed.

With this condition $\mathcal{S}_N$ is of order N and the field integral can be calculated for N large by the steepest descent method. The saddle point values are expected to be $\sigma^2=\mathcal{O}(N)$, $\lambda=\mathcal{O}(1)$.

2.5.3 The saddle point equations

One looks for a uniform saddle point ($\sigma(x),\lambda(x)$ space-independent),

$$\sigma(x)=\sigma\,,\quad \lambda(x)=\lambda\,.$$

The action reduces to

$$\mathcal{S}_N(\lambda,\sigma)/\text{volume}=-\frac{3}{2u}\lambda^2+\frac{3r}{u}\lambda+\frac{1}{2}\lambda\sigma^2+\frac{(N-1)}{2}\int^\Lambda\frac{\mathrm{d}^d p}{(2\pi)^d}\ln\left(p^2+\lambda\right).$$

Differentiating the expression with respect to σ and λ, one obtains the saddle point equations for N large

$$\lambda\sigma = 0\,, \quad \frac{\sigma^2}{N} - \frac{6}{Nu}(\lambda - r) + \frac{1}{(2\pi)^d}\int^\Lambda \frac{\mathrm{d}^d p}{p^2+\lambda} = 0\,.$$

At leading order, a second order phase transition is found at a point $r = r_c$ where $\lambda = \sigma = 0$. For $d > 2$, one finds

$$r = r_c \equiv -\frac{Nu}{6}\frac{1}{(2\pi)^d}\int^\Lambda \frac{\mathrm{d}^d p}{p^2}\,.$$

However, then $\tilde{\Gamma}^{(2)}(p) = p^2$ at leading order. A next order calculation is required to obtain a non-trivial result.

2.5.4 $1/N$ correction

The first non-trivial correction to $\tilde{\Gamma}^{(2)}(p)$ is generated at order $1/N$:

$$\tilde{\Gamma}^{(2)}(p) = p^2 + \frac{2}{N} \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{1}{(6/Nu) + B(q)} \left(\frac{1}{(p+q)^2} - \frac{1}{q^2} \right) + \mathcal{O}\left(\frac{1}{N^2} \right),$$

where $B(q)$ is the one-loop contribution to the perturbative four-point function

$$B(q) = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{k^2 (k+q)^2} \underset{q \to 0}{\sim} b(d) q^{d-4},$$

which is UV finite for $d < 4$. For $d = 3$, $b(1) = 1/8$.

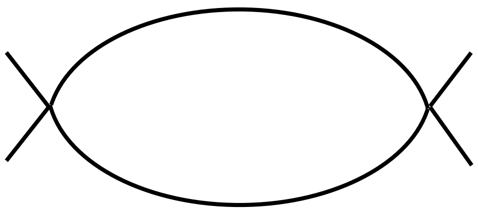

Fig. 2.3 – Feynman diagram: the four-point function $B(p)$.

One evaluates

$$\delta\rho = -\frac{2}{N}\int \frac{\mathrm{d}^d p}{(2\pi)^{2d}} \frac{1}{p^4} \frac{\mathrm{d}^d q}{(6/Nu) + b(d) q^{d-4}} \left(\frac{1}{(p+q)^2} - \frac{1}{q^2} \right)$$

by keeping the dimension d generic and using dimensional regularization. In the $d = 3$ limit, the two integrations yield $(1/32\pi^2)(Nu/6)$.

As expected, $\delta\rho \propto u$:

$$\delta\rho = -u/96\pi^2 = -\frac{a}{\lambda\!\!\!{}^{-}{}^2}.$$

One finally obtains the change in the transition temperature:

$$\frac{\Delta T_c}{T_c} = \frac{8\pi}{3\zeta(3/2)} \frac{a}{\lambda\!\!\!{}^{-}} = c_0 a n^{1/3} \text{ with } c_0 = \frac{8\pi}{3\zeta(3/2)^{4/3}} = 2.33\ldots .$$

Note that, although the final result does not depend on N, the result is only valid for N large.

Taking into account, the $1/N$ correction, one finds $c_0 = 1.71\ldots$. A number of other methods, lattice calculations, summation of perturbative expansions, suggest $c_0 \approx 1.3$.

Conclusions. Using RG arguments, we have shown that the properties of the dilute, weakly interacting Bose gas remain dominated by the UV fixed point up to large length scales; this is why one can still refer to BE condensation when discussing the phase transition of the interacting Bose gas.

RG arguments enabled us to confirm that the relative shift of the transition temperature at fixed density is proportional to the dimensionless combination $an^{1/3}$ for weak interactions. This result is non-perturbative and the proportionality coefficient, which is universal, cannot be obtained from perturbation theory.

Therefore, a non-perturbative method, the large N expansion, has been introduced that allows a systematic, analytic, calculation of this coefficient as a power series in $1/N$, where eventually one has to set $N = 2$.

The leading order contribution is formally of order $1/N$ multiplied by a function of aN, which is kept fixed in the large N limit. Because for $d = 3$ the result is linear in a, the $1/N$ factor somewhat surprisingly cancels and the result is independent of N. Adding the 30% $1/N$ correction, one finds a value in reasonable agreement with more recent numerical estimates.

Lecture 3: THE RELATIVISTIC SCALAR FIELD AT FINITE TEMPERATURE

In simple collider experiments, we mainly probe physics at zero temperature.

Therefore, the study of relativistic Quantum Field Theory (QFT) at finite temperature, was initially motivated by cosmological problems, for example, concerning the restoration of the weak-electromagnetic symmetry in the early universe, but has later gained additional attention in connection with high energy heavy ion collisions and speculations about possible phase transitions in Quantum Chromodynamics (QCD).

In this lecture, we review the temperature dependence of a few properties of Statistical Quantum Field Theory at thermal equilibrium.

First, we point out the relation between quantum field theory in $(1, d)$ dimensions and statistical classical field theory in $(d + 1)$ dimensions.

Finite temperature in the QFT then corresponds to a finite size in one direction of the classical theory.

This identification allows applying renormalization group ideas and the theory of finite size effects of the classical theory to analyse properties of finite temperature QFT.

In particular, we discuss the limit of high temperature (HT) and the situation of finite temperature phase transitions. There dimensional reduction plays an essential role. It reflects the property that quantum effects to some extent are not important at high temperature.

We illustrate these ideas with the example of the ϕ^4 QFT and construct the corresponding effective reduced theory at one-loop order.

3.1 Finite temperature quantum field theory: a few remarks

We discuss QFT at finite temperature in $(1, d)$ dimensions, at equilibrium.

In the case of finite temperature non-relativistic quantum statistical mechanics, quantum effects are important only at low temperature. An important parameter is the ratio between the thermal wave-length $\hbar/\sqrt{mT}$ and the relevant length scale that characterizes the system.

Only when this ratio is large are quantum effects important. Increasing the temperature is at leading order equivalent to decrease $\hbar$.

Moreover, the transition from quantum to classical behaviour is associated with dimensional reduction: the imaginary time disappears.

In a relativistic theory, finite temperature effects are important either because the theory contains massless particles or because one is working in a limit where the temperature T, in energy units, is larger than the rest energy m of massive particles, $T \gg m$ (with the speed of light $c = 1$).

In particular, we want to study the relevance of quantum effects for finite temperature.

We want to understand the conditions under which statistical properties of finite temperature QFT in $(1, d)$ dimension can be described by an effective classical statistical field theory in d dimension.

Since we are interested here only in equilibrium physics, the imaginary time formalism will be used throughout.

3.1.1 Finite temperature scalar quantum field theory

The static properties of finite temperature QFT can be derived from the partition function $\mathcal{Z} = \operatorname{tr} \mathrm{e}^{-H/T}$, where H is the Hamiltonian of the quantum field theory and T the temperature.

For a simple theory with scalar boson fields ϕ and euclidean action $\mathcal{S}(\phi)$, in d space dimensions, the partition function is given by the field integral

$$\mathcal{Z} = \int [\mathrm{d}\phi] \exp\left[-\mathcal{S}(\phi)\right], \tag{3.1}$$

where $\mathcal{S}(\phi)$ is the integral of the Lagrangian density $\mathcal{L}(\phi)$ in imaginary time,

$$\mathcal{S}(\phi) = \int_0^{1/T} \mathrm{d}\tau \int \mathrm{d}^d x \, \mathcal{L}(\phi),$$

and the field ϕ satisfies periodic boundary conditions in the (imaginary) time direction

$$\phi(\tau = 0, x) = \phi(\tau = 1/T, x).$$

3.1.2 Classical statistical field theory and renormalization group

The quantum partition function (3.1) has also the interpretation of the partition function of a classical statistical field theory in $(d+1)$ dimension with a finite size $L = 1/T$ in one direction. The zero temperature limit of the quantum partition function then corresponds to the usual infinite volume classical partition function.

Correlation functions thus satisfy the renormalization group (RG) equations of the corresponding $(d+1)$ dimensional classical field theory.

General results obtained in the study of finite size effects also apply here. RG equations are only sensitive to short distance singularities and, therefore, finite size effects do not modify RG equations.

Finite size effects affect only the solution of the RG equations, because a new dimensionless, RG invariant, variable appears that can be written as the ratio $\xi_L/L \equiv T/m_T$, where the correlation length $\xi_L = 1/m_T$ characterizes the decay of correlation functions and m_T is the finite temperature mass.

For L finite ($T > 0$), we expect a cross-over from a $(d+1)$-dimensional behaviour when the correlation length ξ_L is small compared to L or $m_T \gg T$, to the d-dimensional behaviour when ξ_L is large compared to L or $m_T \ll T$. The latter high temperature regime can be described by an effective d-dimensional theory.

Note that in a quantum field theory the initial microscopic scale Λ^{-1}, where Λ is the QFT cut-off, always appears. Even at high temperature T or $L \to 0$, the product $\Lambda L \equiv \Lambda/T$ remains large.

3.1.3 Mode expansion

As a consequence of periodicity, boson fields can be expanded in Fourier modes in the time direction and the corresponding frequencies are quantized. For boson fields

$$\boldsymbol{\phi}(x,t)=\sum_{\omega_\nu=2\nu\pi T}\mathrm{e}^{i\omega_\nu t}\,\boldsymbol{\phi}_\nu(x),\quad \nu\in\mathbb{Z}\,. \tag{3.2}$$

By contrast, fermion fields $\psi(\tau,x)$ satisfy anti-periodic boundary conditions

$$\psi(\tau=0,x)=-\psi(\tau=1/T,x).$$

Anti-periodic conditions lead to the expansion

$$\boldsymbol{\psi}(x,t)=\sum_{\omega_\nu=(2\nu+1)\pi T}\mathrm{e}^{i\omega_\nu t}\,\boldsymbol{\psi}_\nu(x). \tag{3.3}$$

The absence of a zero-mode, by contrast with bosons, has a major effect at high temperature.

When $T = L^{-1} \gg m$, where m is the zero-temperature physical mass of boson fields, a situation realized at high temperature in the QFT sense, or when the mass vanishes, a non-trivial physics exists for momenta much smaller than the temperature T or distances much larger than L.

In this limit, one expects to be able to treat all non-zero modes as perturbations: the perturbative integration over the non-zero modes leads to an effective field theory for the zero-mode, with a d-dimensional action $\mathcal{S}_T$.

By contrast, due to anti-periodic conditions, fermions have no zero modes. In the same limit fermions can be completely integrated out (but low mass bound states may remain relevant).

Apart from high temperature there is another situation where we expect the mode integration to be useful, in the case of a finite temperature second order phase transition.

Then it is the finite temperature correlation ξ_L that diverges (the mass $m_T = 1/\xi_L$ that vanishes), generating a non-trivial long distance physics.

Remarks.

(i) The mode expansion (3.2, 3.3) is well-suited to simple situations where the field belongs to a linear space. In the case of non-linear σ models or gauge theories the separation of the zero-mode is a more complicated issue.

(ii) The zero-mode has to be treated differently from other modes when $\xi_L = 1/m_T$, the correlation length in the space directions, is large compared to L, equivalently $m_T \ll T$. This condition is equivalent to $m_{T=0} \ll T$ only at leading order in perturbation theory.

3.1.4 Dimensional reduction and effective local field theory

We assume now that we are in the situation where dimensional reduction is relevant.

To construct the effective d-dimensional theory, we thus keep the zero mode and integrate perturbatively over all other modes. It is convenient to introduce the decomposition

$$\phi(x,t) = \varphi(x) + \chi(x,t), \tag{3.4}$$

where φ is the zero mode and χ the sum of all other modes (equation (3.2))

$$\chi(x,t) = \sum_{\nu\neq 0} \mathrm{e}^{i\omega_\nu t}\,\phi_\nu(x), \quad \omega_\nu = 2\pi T\nu\,,\ \nu\in\mathbb{Z}\,.$$

The action $\mathcal{S}_T$ of the reduced theory is defined by

$$\mathrm{e}^{-\mathcal{S}_T(\varphi)} = \int [\mathrm{d}\chi]\exp[-\mathcal{S}(\varphi+\chi)]. \tag{3.5}$$

At leading order in perturbation theory one simply finds

$$\mathcal{S}_T(\varphi) = \frac{1}{T}\int \mathrm{d}^d x\, \mathcal{L}(\varphi).$$

Here $T = 1/L$ plays, in this leading approximation, the formal role of $\hbar$, and the small T, large L, expansion corresponds to a loop expansion.

We are concerned with theories in dimension $d \leq 3$, thus dominated by small momentum (IR) contributions, where the relevant parameter is m/T, which at high temperature is small. Then perturbation theory for the zero mode is no longer possible or useful.

An important parameter in the full effective theory is really m_T/T. Therefore, an important question is whether the integration over non-zero modes, beyond leading order, generates a mass for the zero mode.

3.1.5 Loop corrections to the effective local action

After integration over non-zero modes the effective action contains all possible interactions.

In the high temperature limit one can still perform a local expansion of the effective action.

One expects, but this has to be checked carefully, that in general higher order corrections coming from the mode integration will generate terms which renormalize the terms already present at leading order, and additional interactions suppressed by powers of T.

Exceptions are provided by gauge theories where new low dimensional interactions are generated by the breaking of $O(1,d)$ invariance.

3.1.6 Renormalization

If the initial $(1, d)$ dimensional theory has been renormalized, the complete theory is finite in the formal infinite cut-off limit. However, as a consequence of the zero-mode subtraction, cut-off dependent terms may remain in the reduced d-dimensional action. These terms provide the necessary counter-terms which render the perturbative expansion of the effective field theory finite. The effective can thus be written as

$$\mathcal{S}_T(\varphi) = \mathcal{S}_T^{(0)}(\varphi) + \text{ counter-terms}.$$

Correlation functions have finite expressions in terms of the parameters of the effective action in which counter-terms have been omitted. The first part $\mathcal{S}_T^{(0)}(\varphi)$ thus satisfies the RG equations of the $(d+1)$ dimensional theory.

Finally, the local expansion breaks down at momenta of order T and T plays the role of an intermediate cut-off. Determining the finite parts may involve some careful calculations.

3.1.7 The finite temperature correlation length

As already pointed out, a first and important problem is to understand the behaviour of the effective mass of the zero-mode generated by integrating out the non-zero modes.

If this mass m_T is of order of the QFT temperature $T = L^{-1}$, the zero-mode is no longer different from other modes. The IR problem disappears and one expects to again be able to calculate using standard perturbation theory.

Actually, one should be able to rearrange the $(d+1)$ perturbation theory to treat all modes in the same way.

By contrast, if the mass of the zero-mode remains much smaller than the temperature, perturbation theory is invalidated by IR contributions. However, one can then use the local expansion of the effective action to study the non-perturbative IR properties.

At high temperature the QFT remains with only one explicit scale T. The quantity m_T/T, where m_T is the physical mass of the complete theory, then only depends on dimensionless ratios.

If m_T/T is of order one, the final zero mode acquires a mass comparable to the other modes. Note that this is what happens in theories with non-trivial IR fixed points.

3.2 The example of the $\phi^4_{1,d}$ quantum field theory

We consider an N-component vector $\boldsymbol{\phi}$ and the $O(N)$ symmetric quantum Hamiltonian

$$\mathcal{H}(\boldsymbol{\Pi},\boldsymbol{\phi}) = \tfrac{1}{2}\int \mathrm{d}^d x\, \boldsymbol{\Pi}^2(x) + \Sigma(\boldsymbol{\phi}) \quad \text{with}$$

$$\Sigma(\boldsymbol{\phi}) = \int \mathrm{d}^d x \left\{ \tfrac{1}{2}\left[\nabla\boldsymbol{\phi}(x)\right]^2 + \tfrac{1}{2} r \boldsymbol{\phi}^2(x) + \frac{1}{4!} u \big(\boldsymbol{\phi}^2(x)\big)^2 \right\}, \tag{3.6}$$

a momentum cut-off Λ being implied, to render the field theory UV finite.

The corresponding finite temperature quantum partition function reads

$$\mathcal{Z} = \int [\mathrm{d}\phi] \exp[-\mathcal{S}(\phi)]\,,$$

with periodic boundary conditions in the time direction, $\boldsymbol{\phi}(1/T,x) = \boldsymbol{\phi}(0,x)$ and

$$\mathcal{S}(\phi) = \int_0^{1/T} \mathrm{d}t \left[\int \mathrm{d}^d x\, \tfrac{1}{2}(\mathrm{d}_t\boldsymbol{\phi})^2 + \Sigma(\boldsymbol{\phi}) \right].$$

3.2.1 Phase transitions at zero temperature

Though we are mainly interested in space dimension $d = 3$, it is interesting to also describe the situation in lower dimensions.

As we have pointed out, at zero temperature the quantum model is equivalent to a $(d + 1)$ classical statistical field theory.

Moreover, the quantum field theory is meaningful only if the physical mass m is much smaller than the cut-off Λ and this requires for r to be close to a transition point $r_c(u)$, the famous fine tuning problem.

Dimension $d = 1$*.* For $d = 1$, the field theory for $N = 1$ has an Ising-like phase transition at some special value $r = r_c(u)$ where the mass vanishes (correlation length diverges).

For $N = 2$, the field theory has the peculiar Kosterlitz–Thouless phase transition without ordering and for $N > 2$ no phase transition is possible.

Dimension $d \geq 2$*.* For all N, the field theory has a phase transition at some special value $r = r_c(u, N)$ where the mass vanishes.

3.2.2 Phase transitions at finite temperature

At finite temperature, the quantum field theory has the properties of a classical $(d+1)$ dimensional classical field theory with a finite size in the time direction. Such a theory, from the viewpoint of phase transitions, behaves like a d-dimensional classical theory.

For $d = 1$, no phase transition is possible and the correlation length is bounded.

For $d = 2$, one finds an Ising-like phase transition for $N = 1$, a Kosterlitz–Thouless phase transition without ordering for $N = 2$ and no transition for $N > 2$.

In dimensions $d > 2$, in particular, in the physically relevant dimension $d = 3$, one always finds a phase transition.

More detailed results require an RG analysis with finite size effects and the construction of an effective d-dimensional classical field theory.

3.2.3 Renormalization group at finite temperature

We set

$$r = r_c(u) + \sigma\,,$$

where the quantity $r_c(u)$ has technically the form of a mass renormalization and the deviation σ must satisfy $|\sigma| \ll \Lambda^2$, (a fine tuning from the viewpoint of QFT) for the field theory to be meaningful.

A renormalization group (RG) analysis provides then some useful information. One important quantity is the ratio $m_T/T \equiv L/\xi_L$, where ξ_L is the finite temperature (finite size) correlation length, and m_T, therefore, the mass of the zero-mode in the effective theory.

The zero temperature theory satisfies the RG equations of a $(d+1)$ dimensional field theory in infinite volume. The dimension $d = 3$ is special, because then the $(\phi^2)^2_4$ theory is just renormalizable and IR free (the triviality issue).

Dimension $d = 3$. The theory is just renormalizable and logarithmic deviations from naive scaling appear. RG equations for vertex functions (one-line irreducible in terms of Feynman diagrams) take the form

$$\left[\Lambda\frac{\partial}{\partial\Lambda} + \beta(g)\frac{\partial}{\partial g} - \frac{n}{2}\eta(g) - \eta_2(g)\sigma\frac{\partial}{\partial\sigma}\right]\Gamma^{(n)}(p_i;\sigma,g,\Lambda) = 0\,. \tag{3.7}$$

The dimensionless ratio $m_T/T = F(\Lambda/T, g, \sigma/T^2)$ is RG invariant. Thus it satisfies

$$\left(\Lambda\frac{\partial}{\partial\Lambda} + \beta(g)\frac{\partial}{\partial g} - \eta_2(g)\sigma\frac{\partial}{\partial\sigma}\right)F = 0\,.$$

The solution can be written as (method of characteristics)

$$m_T/T = F(\Lambda/T, g, \sigma/T^2) = F\bigl(\lambda\Lambda/T, g(\lambda), \sigma(\lambda)/T^2\bigr), \tag{3.8}$$

where λ is a scale parameter, and $g(\lambda), \sigma(\lambda)$ the effective parameters at scale λ),

$$\lambda\frac{\mathrm{d}g(\lambda)}{\mathrm{d}\lambda} = \beta\bigl(g(\lambda)\bigr), \quad \lambda\frac{\mathrm{d}\sigma(\lambda)}{\mathrm{d}\lambda} = -\sigma(\lambda)\eta_2\bigl(g(\lambda)\bigr).$$

The form of the RG β-function implies that the theory is IR free since,

$$\beta(g) = \frac{(N+8)}{48\pi^2} g^2 + O\left(g^3\right),$$

and, thus, the effective coupling constant at the physical scale is logarithmically small: $g(\lambda) = O(1/\ln\lambda)$ for $\lambda \to 0$.

For example, to reach the scale $T = 1/L$ we choose $\lambda = T/\Lambda \ll 1$ and, thus,

$$g(T/\Lambda) \sim \frac{48\pi^2}{(N+8)\ln(\Lambda/T)}. \tag{3.9}$$

From

$$\eta_2(g) = -\frac{N+2}{48\pi^2} g + O\left(g^2\right),$$

one also finds

$$\sigma(T/\Lambda) \propto \frac{\sigma}{(\ln\Lambda/T)^{(N+2)/(N+8)}}. \tag{3.10}$$

Therefore, RG improved perturbation theory can be used.

Dimensions $d = 2$. The three-dimensional classical theory has an IR fixed point $g^* > 0$. Finite size scaling (equation (3.8)) predicts, in the symmetric phase,

$$m_T/T = f(\sigma T^{-1/\nu}),$$

where ν is the $d = 3$ correlation exponent. Therefore, $m_T = O(T)$ is large except for $f(x) \ll 1$.

This occurs near a continuous phase transition, but for $d = 2$ such a transition is possible only for $N = 1$ and the system at zero temperature is in a broken phase ($N = 2$ in the massless phase must be examined separately).

If $f(x = x_0) = 0$, then $m_T/T \ll 1$ for

$$|x - x_0| \ll 1 \Rightarrow |T - T_c| \ll (-\sigma)^\nu \text{ with } T_c = x_0^{-\nu}(-\sigma)^\nu \propto m_{T=0} \propto T\,,$$

and the IR properties are described by an effective two-dimensional theory.

The situation of a massless phase at zero temperature for $N \neq 1$ must be examined separately.

3.3 One-loop effective action

Mode expansion and effective action at leading order. To construct the effective, d-dimensional field theory, one expands the field in Fourier modes in the time direction (equation (3.2)).

One integrates perturbatively over all non-zero modes (equation (3.5)). In the notation (3.6) the result at leading order simply is

$$\mathcal{S}_T(\boldsymbol{\varphi}) = \frac{1}{T}\,\Sigma(\boldsymbol{\varphi}).$$

After the rescaling $\boldsymbol{\varphi}$ into $\boldsymbol{\varphi}T^{1/2}$, the action becomes

$$\mathcal{S}_T(\boldsymbol{\varphi}) = \int \mathrm{d}^d x \left\{ \tfrac{1}{2}\left[\nabla\boldsymbol{\varphi}(x)\right]^2 + \tfrac{1}{2}\sigma\boldsymbol{\varphi}^2(x) + \frac{1}{4!}uT\left(\boldsymbol{\varphi}^2(x)\right)^2 \right\}. \qquad (3.11)$$

At this order r_c vanishes and, therefore, has been omitted.

In terms of the dimensionless bare coupling $g = u\Lambda^{d-3}$, the expansion parameter is $(\Lambda/m_T)^{4-d}(T/\Lambda)g$.

For $d = 3$, the expansion parameter reduces to gT/m_T. Since dimensional reduction is useful only for m_T/T small, the situation is subtle because the running coupling constant $g(T/\Lambda)$ at scale $T \ll \Lambda$, renormalized by higher modes, is also small: $g(T/\Lambda) = O(1/\ln(\Lambda/T))$.

A more detailed discussion of the situation requires a one-loop calculation.

For $d < 3$, IR singularities are always present both in the initial and the reduced theory, and the small coupling regime can never be reached for interesting situations.

The $\varepsilon = 3 - d$ expansion can be useful in some limits, otherwise the problem has to be studied by non-perturbative methods.

3.3.1 One-loop calculation

The one-loop contribution to the effective action, generated by integrating over the non-zero modes, is ($\ln\det = \operatorname{tr}\ln$)

$$\mathcal{S}_T^{(1)}(\boldsymbol{\varphi}) = \tfrac{1}{2}\operatorname{tr}\ln\left[(-\mathrm{d}_t^2 - \nabla_x^2 + \sigma + \tfrac{1}{6}u\boldsymbol{\varphi}^2)\delta_{ij} + \tfrac{1}{3}u\varphi_i\varphi_j\right] - \ (\boldsymbol{\varphi} = 0).$$

The situation of interest here is when the physical mass m at zero temperature (the inverse of the correlation length $\xi = 1/m$ of the infinite volume $(d+1)$-dimensional system) is smaller than the temperature T.

For $m < T$, a local expansion in $\boldsymbol{\varphi}$ is justified.

The leading order in the derivative expansion is obtained by treating $\boldsymbol{\varphi}(x)$ as a constant.

The evaluation of the one-loop contribution $\mathcal{S}_T^{(1)}$ to the reduced action, in the Fourier representation at fixed momentum reduced to the calculation of the partition function of the harmonic oscillator.

This yields the identity

$$\operatorname{tr}\ln(-\mathrm{d}_t^2 - \nabla_x^2 + M^2) = V_d \int \frac{\mathrm{d}^d k}{(2\pi)^d} \sum_{n\neq 0} \ln(\omega_n^2 + k^2 + M^2)$$
$$= 2V_d \int \frac{\mathrm{d}^d k}{(2\pi)^d} \ln\left[2\sinh\big(L\omega(k)/2\big)/\omega(k)\right], \tag{3.12}$$

where V_d is the d volume and $\omega(k) = \sqrt{k^2 + M^2}$.

Applied to the present example, one finds

$$\mathcal{S}_T^{(1)}(\boldsymbol{\varphi}) = \int \mathrm{d}^d x \int \frac{\mathrm{d}^d k}{(2\pi)^d} \left\{(N-1)\ln\left[2\sinh\big(L\omega_{\mathrm{T}}(k)/2\big)/\omega_{\mathrm{T}}(k)\right]\right.$$
$$\left. + \ln\left[2\sinh\big(L\omega_{\mathrm{L}}(k)/2\big)\omega_{\mathrm{L}}(k)\right]\right\} - (\boldsymbol{\varphi} = 0) \tag{3.13}$$

with

$$\omega_{\mathrm{T}}(k) = \sqrt{k^2 + \sigma + \tfrac{1}{6}u\boldsymbol{\varphi}^2(x)}, \quad \omega_{\mathrm{L}}(k) = \sqrt{k^2 + \sigma + \tfrac{1}{2}u\boldsymbol{\varphi}^2(x)}.$$

3.3.2 φ-expansion

At this order, an expansion in powers of φ makes sense only if $-\sigma/T^2 < 4\pi^2$, a condition that constrains σ only for $\sigma < 0$ and more generally involves the dimensionless RG invariant ratio $m_T/T = L/\xi_L$, where m_T is the mass parameter of the ordered phase. The condition implies that the expansion around $\varphi = 0$ is meaningful only $\langle\varphi\rangle$ is small enough.

Order φ^2. For the quadratic term the local approximation is not needed because the corresponding one-loop diagram is a constant:

$$\left[\mathcal{S}_T^{(1)}\right]_2 = \frac{1}{12}(N+2)\bar{G}_2(\sigma,T)\frac{u}{T}\int \mathrm{d}^d x\, \boldsymbol{\varphi}^2(x)\,,$$

where

$$\bar{G}_2(\sigma,T) = \int \frac{\mathrm{d}^d k}{(2\pi)^d \omega(k)}\left(\frac{1}{2} + \frac{1}{\mathrm{e}^{\omega(k)/T} - 1} - \frac{T}{\omega(k)}\right) \tag{3.14}$$

with now $\omega(k) = \sqrt{k^2+\sigma}$. One recognizes the sum of the zero-temperature result, the thermal fluctuations and the subtracted zero-mode contribution.

Introducing the zero-temperature one-loop diagram,

$$\Omega_{d+1}(m) = \frac{1}{(2\pi)^{d+1}} \int^{\Lambda} \frac{\mathrm{d}^{d+1}k}{k^2 + m^2}\,, \tag{3.15}$$

and the UV finite function

$$\begin{aligned} f_d(s) &= N_d \int_0^\infty \frac{x^{d-1}\mathrm{d}x}{\sqrt{x^2+s}} \frac{1}{\exp[\sqrt{x^2+s}]-1} \\ &= N_d \int_{\sqrt{s}}^\infty (y^2 - s)^{(d-2)/2} \frac{\mathrm{d}y}{\mathrm{e}^y - 1}\,, \end{aligned} \tag{3.16}$$

where N_d is the loop factor,

$$N_d = \frac{2}{(4\pi)^{d/2}\Gamma(d/2)}, \tag{3.17}$$

we can rewrite $\bar{G}_2$ as

$$\bar{G}_2(\sigma, T) = \Omega_d(\sqrt{\sigma}) - T\Omega_d(\sqrt{\sigma}) + T^{d-2} f_d(\sigma/T^2). \tag{3.18}$$

In particular,

$$f_d(0) = N_d\Gamma(d-1)\zeta(d-1), \quad f'_d(0) = -\tfrac{1}{2}(d-2)N_d\Gamma(d-3)\zeta(d-3), \quad (3.19)$$

where $\zeta(s)$ is Riemann's ζ function.

Order $(\boldsymbol{\varphi}^2)^2$. The quartic term is proportional to the initial interaction

$$\left[\mathcal{S}_T^{(1)}\right]_4 = -\frac{1}{144}(N+8)\bar{G}_4(\sigma,T)\frac{u^2}{T}\int \mathrm{d}^d x\,(\boldsymbol{\varphi}^2(x))^2.$$

One verifies

$$\bar{G}_4(\sigma,T) = -\frac{\partial}{\partial\sigma}\bar{G}_2(\sigma,T)\,. \quad (3.20)$$

The one-loop reduced action. We first keep only the terms already present in the tree approximation. The coefficient $r = r_c$ of ϕ^2 is defined by the condition that at $T = 0$ the mass vanishes. Thus, $\bar{G}_2$ has to be replaced by $(d > 1)$

$$[\bar{G}_2]_{\rm r}(\sigma, T) = \bar{G}_2(\sigma, T) - \Omega_{d+1}(0)$$
$$= -\sigma D_{d+1}(\sqrt{\sigma}) - T\Omega_d(\sqrt{\sigma}) + T^{d-1} f_d(\sigma/T^2)$$

with

$$D_d(m) = \frac{1}{(2\pi)^d} \int^{\Lambda} \frac{\mathrm{d}^d k}{k^2(k^2 + m^2)} = \frac{1}{m^2} \left[\Omega_d(0) - \Omega_d(m)\right]. \tag{3.21}$$

After the rescaling $\varphi \mapsto \varphi T^{1/2}$, the effective action can be written as

$$\mathcal{S}_T(\varphi) = \int \mathrm{d}^d x \left\{ \frac{1}{2} \left[\nabla \varphi(x)\right]^2 + \frac{1}{2} r_1 \varphi^2(x) + \frac{1}{4!} T u_1 \left(\varphi^2(x)\right)^2 \right\} \tag{3.22}$$

with

$$r_1 = \sigma + \tfrac{1}{6}(N+2) u [\bar{G}_2]_{\rm r}\,, \quad u_1 = u - \tfrac{1}{6}(N+8) u^2 \bar{G}_4\,.$$

Other interactions. For space dimensions $d < 6$, the coefficients of the other interaction terms are no longer UV divergent. Since the zero-mode contribution has been subtracted no IR divergence is generated even in the massless limit. In this limit, the coefficients are thus proportional to powers of $1/T$ obtained by dimensional analysis (in the normalization (3.22)):

$$[\mathcal{S}_T]_{(2n)} \propto g^n (T/\Lambda)^{n(d-3)} T^{d-n(d-2)} \int \mathrm{d}^d x \, (\boldsymbol{\varphi}^2(x))^n ,$$

and, therefore, increasingly negligible at high temperature at least for $d \geq 3$.

The local expansion of the one-loop determinant also generates monomials with derivatives. No term proportional to $(\nabla\varphi)^2$ is generated at one-loop order. All other terms with derivatives are finite for $d < 6$, and thus the structure of the coefficients again is given by dimensional analysis. To $2k$ derivatives corresponds an additional factor $1/T^{2k}$.

Finally, for $\sigma \neq 0$ but $\sigma/T^2 \ll 1$, we can expand in powers of σ and the previous arguments immediately generalize.